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Equations of transfer in non-local media

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Abstract—The transfer equations and Fourier law analogues have been obtained for three types of non-local media: media with heat memory, spatially non-local media and media with a discrete structure. The conditions are specified under which these equations reduce into each other or to familiar transfer equations, such as the classical parabolic-type transport equation and 'telegraph' equation. It is shown that the type of partial differential equations derived from discrete transfer equations is governed by the limiting transition law, i.e. by the relationship between the time, τ, and space, h, scales of the medium internal structure. In the case of the 'diffusional' law of limiting transition, when the thermal diffusivity coefficient $a = h^2/4\tau = \text{const}$ for τ, $h \to 0$, the discrete equations yield parabolic-type partial differential equations, whereas with the 'wave' law of limiting transition, when the heat wave speed $c = h/2\tau = \text{const} < \infty$ for τ, $h \to 0$, they yield partial differential equations of hyperbolic type.

1. INTRODUCTION

CLASSICAL thermodynamics, like continuum mechanics, is based on the principle of locality. According to this principle, the fundamental laws of thermodynamics hold not only for the body considered as a whole, but also for each, however small, of its parts [1,2]. The locality principle implies that such a medium does not have its own internal structure, and this allows one to perform limiting transition to an infinitesimal volume when passing from the integral to the most practical differential form of conservation laws. Moreover, classical non-equilibrium thermodynamics uses the principle of local thermodynamic equilibrium which means that the entire system can be divided into fairly small, but still macroscopic areas, which represent equilibrium thermodynamic systems. Consequently, physically infinitesimal areas of a totally non-equilibrium system are described in the same way as equilibrium systems [1, 2], i.e. locally equilibrium systems can be treated as local in time.

Classical thermodynamics, based on the space–time locality principles, rests on the Fourier law

$$\vec{q}(\vec{X}, t) = -\lambda \nabla T(\vec{X}, t),$$

where $\vec{q}$ is the heat flux, T the temperature, and λ the heat conduction coefficient. The Fourier law and the energy conservation law in differential form yield a transfer equation of parabolic type. It is obvious that this equation adequately describes transfer processes only under the conditions when the approximations of local thermodynamic equilibrium and spatial locality are fulfiled. Otherwise, the use of parabolic-type transfer equations can lead to false results. For example, it is known that this equation furnishes a physically incorrect conclusion about an infinitely large speed of propagation of a heat wave (heat signal), since a sharp change in temperature at a certain space point is instantaneously perceived even at an infinitely distant point [2, 5]. In real systems this speed has a finite, although still significant, value. Because of this for fast processes, the characteristic speed of which is comparable with the speed of heat wave propagation, the classical theory of transfer processes cannot be applied. In addition, the classical approach does not always adequately describe transfer processes in systems with a complex heterogeneous structure, such as polymers, capillary-porous media, liquid crystals, suspensions, pastes, etc. In this case, the principle of spatial locality can be violated when the characteristic scale of the process of transfer becomes comparable with that of the internal structure of the medium. A similar situation occurs in plasma [12] or in a solid body [13] exposed to laser radiation where the electron or phonon-free path plays the role of the scale of internal structure (non-locality scale).

There exist a number of versions of locally non-equilibrium theories, i.e. the theories which do not rest upon the principle of local equilibrium. Among these, the most developed and consistent seem to be the 'extended irreversible thermodynamics' [2, 3] and 'rational thermodynamics' [4]. Moreover, there are many works in the literature that employ various kinetic, thermodynamic, phenomenological, and some other methods for describing locally non-equilibrium systems; references to which can be found in refs. [1–11]. In the study of spatial non-locality [12, 13] and space–time discreteness [6,14,15] of transfer processes only the initial steps have been undertaken.

Recently, in the different fields of physics, chem-

NOMENCLATURE

A	constant value
$a = h^2/4\tau$	thermal diffusivity
$c = h/2\tau$	speed of heat wave
C_p	heat capacity
E	internal energy
h	characteristic scale of spatial non-locality
j	heat flux non-locality function
$K(z)$	heat flux relaxation function
L	macroscale
$\bar{q}$	heat flux
T	temperature
t	time
V	characteristic velocity (e.g. velocity of heat source motion)
W	heat source intensity
$\bar{X}$	coordinate.

Greek symbols

$\beta(z)$	internal energy relaxation function
γ	internal energy non-locality function
λ	heat conduction coefficient
ρ	density
τ	time of relaxation to local thermodynamic equilibrium.

istry, biology and the like, a great number of non-local systems have been discovered [2, 5–7, 12, 13], the description of which requires appropriate transfer equations. On the one hand, such non-local equations should have the simplest possible form convenient for specific calculations, and, on the other, they should take into account the main specific features of the given system. Thus, for example, very often the non-local theories do not take into consideration the presence in the system of heat (mass) sources which in real systems exert a decisive influence on transfer processes. The present work considers equations of transfer in three types of non-local media which are most widespread in practice: locally non-equilibrium media (or media with memory), spatially non-local media and discrete-structure media.

For such systems the equations of transfer are obtained in a general form; certain simplest particular cases are considered, and conditions of reduction to classical local equations are specified. Wherever possible the inter-relation between different types of non-local equations is shown.

2. LIMITS OF APPLICABILITY OF THE LOCAL THEORIES OF TRANSFER PROCESSES

The approximation of local thermodynamic equilibrium implies that in each small medium element there exists the condition of local equilibrium which can be described by a local equilibrium distribution function. The entropy of such a condition is the same function of macroscopic variables as for an equilibrium system [1, 2]. The local thermodynamic equilibrium can develop in a system the rate of variation of its macroparameters due to external effects, chemical reactions, phase transition, etc. (i.e. the rate of equilibrium violation) which is much smaller than the rate of equilibrium establishment in a local volume. Moreover, the local thermodynamic equilibrium approximation holds for the time moments, t, significantly exceeding the characteristic time of system relaxation towards the local equilibrium τ^* [1]. These conditions can be written in the following form:

$$T/(\partial T/\partial X) \gg h_r \tag{1}$$

$$T/(\partial T/\partial t) \gg \tau^*, \tag{2}$$

where h_r is the characteristic relaxation length, i.e. the scale of the processes of local equilibrium establishment (for gases, this is the mean free path of molecules).

As noted above, it follows from the locality principle that the system does not have its own internal discrete structure and it can be considered as the continuum. Strictly speaking, from the physical point of view, this statement is inaccurate, since a medium always consists of separate elements: atoms, molecules, cells or some other more complex subsystems. However, if the characteristic macroscale of the system is much in excess of the characteristic size of its microstructure h_s, i.e.

$$L \gg h_s \tag{3}$$

then the discrete structure of this system may be neglected and it can be considered to be a continuous (local) medium. For gases, the value h_s is of the order of the free path of molecules. In this case inequality (3) is equivalent to inequality (1), and the local thermodynamic equilibrium approximation is equivalent to that of continuum (generally speaking, the reverse can be incorrect). Such a correspondence will be observed in other systems if $h_r \approx h_s$.

Let us consider inequalities (1)–(3) for the case when a heat source moves in a medium with a constant velocity V. Here the temperature profile is formed as a travelling wave with a characteristic front width $L = a/V$ (where a is the thermal diffusivity) and characteristic time $t^* = a/V^2$ [6, 7]. Then inequalities (1)–(3) will acquire the form [6, 7]

$$a/V \gg h, \qquad a/V^2 \gg \tau^*. \tag{4}$$

Taking into account the fact that τ^* is related to the speed of heat wave (thermal disturbances) $c = (a/\tau^*)^{1/2}$ [2, 5] and that $h = \tau^* c$, it will be shown that inequalities (4) are equivalent to the inequality

$$V \ll c.$$

Thus, the locality and local thermodynamic equilibrium principles are valid if the speed of the heat source motion (the characteristic speed of the macroprocess) V is much smaller than that of the heat wave (heat signal) C. In other words, in local theories, an infinite speed of heat wave propagation is assumed, which corresponds to the above-mentioned 'paradox of instantaneous heat propagation'. The non-local theories that take into account the finite speed of heat wave (i.e. thermal disturbances) can be considered in this sense as relativistic with respect to the classic theories.

3. MEDIA WITH HEAT MEMORY

One of the most general models of locally non-equilibrium media with a finite speed of heat wave propagation involve the media with heat memory [1, 4–11]. In such media the heat flux $\bar{q}(\bar{X}, t)$ and the internal energy $E(\bar{X}, t)$ depend not on instantaneous values of the gradient and temperature, as in equilibrium media, but are determined by the whole prehistory of the heat transfer in the considered medium element (such media can also be called non-local in time)

$$\bar{q}(\bar{X}, t) = -\int_0^\infty K(z)\,\nabla T(\bar{X}, t-z)\,\mathrm{d}z \tag{5}$$

$$E(\bar{X}, t) = C_p \rho T(\bar{X}, t) + \int_0^\infty \beta(z) T(\bar{X}, t-z)\,\mathrm{d}z, \tag{6}$$

where $K(z)$ and $\beta(z)$ are the relaxation functions of heat flux and internal energy, respectively. The heat flux and internal energy of the system are inter-related by the law of energy conservation:

$$\frac{\partial E}{\partial t} = -\operatorname{div} \bar{q}(\bar{X}, t) + W(\bar{X}, t), \tag{7}$$

where $W(\bar{X}, t)$ is the intensity of energy sources distributed throughout the system. Equations (5)–(7) yield several versions of the heat conduction equations for media with heat memory that correspond to various thermodynamic restrictions [1, 4–11]. In ref. [9] the hyperbolic-type heat conduction equation with a finite speed of heat wave (heat signal) propagation, $c = (K(0)/C_p\rho)^{1/2}$ is considered. In ref. [10] the dependence of heat flux on instantaneous temperature gradient in relation (5) was isolated in explicit form, and, based on this, a parabolic-type equation of heat conduction were obtained. In refs. [6, 7] heat conduction equations (5)–(7) were suggested for a medium with memory which represents a combination of two above-mentioned equations:

$$\tau^* C_p \rho \frac{\partial^2 T}{\partial t^2} + \int_0^\infty [\tau^* K'(z) + K(z)]\,\Delta T(\bar{X}, t-z)\,\mathrm{d}z + \left(C_p\rho + \tau^*\beta(0)\right)\frac{\partial T}{\partial t} = \int_0^\infty \left(\tau^*\beta'(z) + \beta(z)\right)\frac{\partial T(\bar{X}, t-z)}{\partial t}\,\mathrm{d}z + \tau^* K(0)\,\Delta T + W + \tau^*\frac{\partial W}{\partial t}. \tag{8}$$

Equation (8) is of hyperbolic type, which reflects the wave character of heat propagation in media with memory. Equation (8) involves both the source function itself $W(\bar{X}, t)$ and also its derivative $\partial W/\partial t$, which is the consequence of the inertia properties of heat conduction in such media.

With the exponential function of heat flux relaxation

$$K(z) = K(0)\exp(-z/\tau^*) \tag{9}$$

relation (5) corresponds to the Maxwell–Cattaneo law [2, 5]:

$$\bar{q}(\bar{X}, t) + \tau^* \frac{\partial \bar{q}(\bar{X}, t)}{\partial t} = -\lambda \nabla T(\bar{X}, t), \tag{10}$$

where $\lambda = \tau^* K(0)$ is the thermal conductivity coefficient. The characteristic time of heat flux relaxation τ^* in equation (9) corresponds to the time of system relaxation towards the local equilibrium in equation (10). Assuming further that the internal energy relaxation function $\beta(z) \equiv 0$, we shall obtain from equation (8) a hyperbolic-type heat conduction equation, the so-called 'telegraph' equation [2, 5, 7]:

$$C_p\rho \frac{\partial T}{\partial t} + \tau^* C_p \rho \frac{\partial^2 T}{\partial t^2} = \lambda \Delta T + W + \tau^* \frac{\partial W}{\partial t}. \tag{11}$$

This equation, just like with the generalized equation (8), involves both diffusional properties of the heat conduction process (energy dissipation) and wave properties (heat wave propagation with a finite speed). When $\tau^* \to 0$ equation (11) yields the classical locally equilibrium parabolic-type heat conduction equation (diffusion equation) which can also be obtained from equation (8) if $\tau^* \to 0$ and $K(0) \to \infty$ so that $\lambda = \tau^* K(0) = \text{const} > 0$. In this case $K(z) \to \lambda\delta(z)$, where $\delta(z)$ is the Dirac delta function, and relation (5) is reduced to the classical Fourier law. Thus, a hyperbolic-type heat conduction equation (11) ('telegraph' equation) which describes the space–time temperature distribution in locally non-equilibrium systems, can be obtained on the basis of various approaches (see refs. [1–7] and the references therein). This indicates its universality and possible wide range of practical application [2, 5–7].

4. NON-LOCAL MEDIA

In general, the processes of transfer are non-local by their nature, i.e. non-local energy (or substance) is transferred from one point of space to another. If the characteristic scale of the internal structure of a heterogeneous system or the mean free path of the particles transporting the energy (of phonons and electrons) becomes commensurable with the scale of the transfer process, then the non-local effects become significant, and this should be taken into account when describing the dynamics of such systems. In this case, the heat flux at a given point of space depends on the temperature gradient throughout the entire system [12, 13]:

$$\bar{q}(\bar{X},t) = -\int_v j(\bar{X}-\bar{\xi})\,\nabla T(\bar{\xi},t)\,\mathrm{d}v, \qquad (12)$$

where j is the heat flux non-locality function. In other words the non-local media described by equation (12) can be considered as media with spatial memory [see equation (5)]. If the system possesses both spatial and temporal memory, then the expressions for the heat flux and internal energy are of the following form [6]:

$$\bar{q}(\bar{X},t) = -\int_v\int_0^\infty j(\bar{X}-\bar{\xi},z)\nabla T(\bar{\xi},t-z)\,\mathrm{d}v\,\mathrm{d}z$$

$$E(\bar{X},t) = C_p\rho T(\bar{X},t) + \int_v\int_0^\infty \gamma(\bar{X}-\bar{\xi},z)T(\bar{\xi},t-z)\,\mathrm{d}v\,\mathrm{d}z.$$

Further, consider that the system is non-local only in space. Then, by analogy with equation (16), the following relation can be written (one of the simplest cases of space–time non-locality is considered in Section 5):

$$E(\bar{X},t) = C_p\rho T(\bar{X},t) + \int_v \gamma(\bar{X}-\bar{\xi})T(\bar{\xi},t)\,\mathrm{d}v. \qquad (13)$$

The heat flux, $\bar{q}$, and internal energy, E, are interrelated by the energy conservation law. Since the differential energy conservation equation (7) was obtained in the local approximation, then in non-local media one should use an integral form of the conservation law:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v E(\bar{X},t)\,\mathrm{d}v = \int_s \bar{q}(\bar{X},t)\,\mathrm{d}\bar{S} + \int_v W(\bar{X},t)\,\mathrm{d}v. \qquad (14)$$

From equations (13) and (14) we obtain a non-local transfer equation in the general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_v \left[C_p\rho T + \int_v \gamma(\bar{X}-\bar{\xi})T(\bar{\xi},t)\right]\mathrm{d}v = \int_s\int_v j(\bar{X}-\bar{\xi})\nabla T(\bar{\xi},t)\,\mathrm{d}v\,\mathrm{d}\bar{S} + \int_v W(\bar{X},t)\,\mathrm{d}v.$$

The form of the corresponding non-locality functions is determined by specific features of the heat transfer process in a given particular system.

4.1. *Example*

As an example consider a one-dimensional case assuming that the non-locality function of the heat flux j is even [12, 13], i.e. the effect of non-locality on the heat transfer process is symmetric about the given point of space. Then, from equation (12) we obtain

$$q(X,t) = \int_0^\infty j(z)\frac{\partial}{\partial z}(T(X-z,t) - T(X+z,t))\,\mathrm{d}z.$$

From the fundamental postulates of thermodynamics it follows that the spatial memory, like the memory in time [equation (9)], should be of damping character [1]. Consequently, it may be considered that [12]

$$j(z) = A\exp(-z/h). \qquad (15)$$

Here h is the characteristic non-locality scale and $A = \text{const}$. After some transformations, we obtain that at $2hA = \lambda$

$$q(X,t) = -\lambda\frac{\partial T}{\partial X} - \lambda h^2\frac{\partial^3 T}{\partial X^3} - \ldots. \qquad (16)$$

In a local medium, when $h = 0$, relation (16) is reduced to the classical Fourier law. The second and subsequent terms on the right-hand side of equation (16) represent a correction to the Fourier law to take into account the spatial non-locality of transfer processes.

In addition, the spatial non-locality of the transfer process leads to an alternative differential form to represent the energy conservation law in comparison with the classical form of equation (7). In fact, in the considered case, the integration volume in the integral conservation law (14) cannot be allowed to approach zero, because it is limited from below by the volume with the characteristic linear dimension of the order of the non-locality length h. In a one-dimensional case, it follows from equation (14) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x-h}^{x+h} E(X,t)\,\mathrm{d}X = q(X-h,t) - q(X+h,t) + \int_{x-h}^{x+h} W(X,t)\,\mathrm{d}X.$$

The Taylor expansion in terms of the powers h yields

$$\frac{\partial}{\partial t}\left(E + \frac{h^2}{6}\frac{\partial^2 E}{\partial X^2} + \ldots\right) = -\frac{\partial q}{\partial X} - \frac{h^2}{6}\frac{\partial^3 q}{\partial X^3} - \ldots + W + \frac{h^2}{6}\frac{\partial^2 W}{\partial X^2}. \qquad (17)$$

Expression (17) represents a differential form of the energy conservation law for a non-local medium with the characteristic scale h. When $h \to 0$, equation (17) is reduced to the classical form of the local conservation law (7). Excluding the heat flux q from equations (16) and (17) gives a heat conduction equation for a non-local medium with an exponential relaxation function (15):

$$C_p\rho\frac{\partial T}{\partial t}+C_p\rho h^2\frac{\partial^3 T}{\partial t\,\partial X^2}=\lambda\frac{\partial^2 T}{\partial X^2}+\frac{7}{6}\lambda h^2\frac{\partial^4 T}{\partial X^4}+W+\frac{h^2}{6}\frac{\partial^2 W}{\partial X^2}+O(h^4). \quad (18)$$

Thus, to describe non-local systems, the energy (mass) conservation law should be used in a non-local integral [equation (14)], or differential [equation (17)] form. With the exponential non-locality function (15), the analogue of the Fourier law (16) and heat conduction equation (18) contain higher derivatives of T and W in comparison with the local case.

5. MEDIA WITH DISCRETE STRUCTURES

The systems which are non-local both in time and in space represent natural generalizations of the models of non-local systems considered in the preceding sections. The space–time non-locality originates when the time of system relaxation to the local thermodynamic equilibrium τ^* becomes commensurable with the characteristic time of the transfer process t^*, whereas the microscale of the internal structure of the medium h commensurable with the characteristic macroscale L of the process (see Section 2). The simplest examples of systems with space–time non-locality are furnished by media with discrete structures, in which the processes of transfer can be described by the random walk model (see refs. [6, 7, 14, 15] and the references therein). Based on this model, the authors in refs. [14, 15] suggested equations of heat conduction in inert media (i.e. in media without heat sources). Of greatest interest for the theory of transfer processes and for its practical application are active media with distributed energy sources. Moreover, in refs. [14, 15] various versions of the limiting transition from the discrete transfer equation to its approximations of different accuracy in the form of partial differential equations were not taken into account. As it will be shown, the specific features of such a limiting transition determine the type of partial differential equations, and, consequently, certain fundamental properties of their solutions. To obtain a discrete transfer equation in a non-local active medium use will be made of the random walk model supplemented with the concepts of energy sources distributed in a system [6, 7].

Consider a two-dimensional heat transfer process in a medium consisting of particles performing random walks. The distance over which a particle (phonon) carries energy for a single act of walking h is the microscale, characterizing the discrete spatial structure of heat transfer. The time between two successive acts of walking τ is the scale of the time discreteness of heat transfer (it will be shown below that $\tau/2 = \tau^*$ can be considered as the time of system relaxation to local thermodynamic equilibrium). The probabilities of the particle motion in four possible directions will be denoted as P_1, P_2, P_3, and P_4, with $P_1+P_2+P_3+P_4 = 1$. If the probability that the particle at time $t+\tau$ will be found within a certain element of the medium $\bar{X}$ is $u(\bar{X}, t+\tau)$, then

$$u(\bar{X},t+\tau) = P_1u(X_1+h,X_2,t)+P_2u(X_1-h,X_2,t)+P_3u(X_1,X_2+h,t)+P_4u(X_1,X_2-h,t). \quad (19)$$

Equality (19) means that for one time step the particle can get into the medium element, $\bar{X}$, only from adjacent elements. Assuming further that the local internal energy of the medium is proportional to the number of particles (phonons) in a given discrete medium element [14, 15] and to the intensity of the distributed energy sources $F(\bar{X}, t)$ [6, 7], equation (19) will yield

$$T(\bar{X},t+\tau) = P_1T(X_1+h,X_2,t)+P_2T(X_1-h,X_2,t)+P_3T(X_1,X_2+h,t)+P_4T(X_1,X_2-h,t)+F(\bar{X},t). \quad (20)$$

Expression (20) represents a discrete heat transfer equation with distributed energy sources. Note that the discrete structure of this equation determines the finite propagation speed of thermal disturbances (heat wave) $c = h/\tau$. Based on the random walk model, we can also obtain the relationship between the heat flux $\bar{q}(\bar{X}, t)$ and temperature $T(\bar{X}, t)$ in a discrete form (the discrete analogue of the Fourier law):

$$\bar{q}\left(\bar{X},t+\frac{\tau}{2}\right) = -\frac{hC_p\rho}{4\tau}\begin{pmatrix}T(X_1+h/2,X_2,t)-T(X_1-h/2,X_2,t)\\ T(X_1,X_2+h/2,t)-T(X_1,X_2-h/2,t)\end{pmatrix}. \quad (21)$$

From now on it will be assumed for simplicity that the medium is isotropic, i.e., $P_1 = P_2 = P_3 = P_4 = 1/4$. It follows from physical considerations that heat flux in a medium with discrete time and space represents an amount of heat energy carried per unit of time through a unit surface of the considered medium element between two successive time moments t and $t+\tau$ when the system was in a state with a certain temperature. Therefore, in relation (21) the heat flux $\bar{q}(\bar{X}, t+\tau/2)$ is presented at the intermediate instant of time $t+\tau/2$, and the temperature in the right-hand side of equation (21) is taken at the boundaries of the given element at the points $X+h/2$ and $X-h/2$. Note that the latter circumstance is insignificant when analysing transfer processes in discrete form, since, by discrete model definition, the temperature over the interval of length h is constant, i.e. $T(X_1 \pm h/2, X_2, t) = T(X_1 \pm h, X_2, t)$, $T(X_1, X_2 \pm h/2, t) = T(X_1, X_2 \pm h, t)$. The shift by $h/2$ is required for limiting transition from a discrete form of transfer equations to partial differential equations, as done below. For this purpose it is necessary to couple the heat function in discrete equation (24) $F(\bar{X}, t)$ with the heat function corresponding to the differential transfer equation. First, it should be noted that the theories of coupled maps [16] and cellular automata [17] rely on the following discrete equation, which is similar in form to equation (20):

$$\theta(\bar{X}, t+\tau) = \frac{D}{4}[\theta(X_1+h, X_2 t)+\theta(X_1-h, X_2, t)$$

$$+\theta(X_1, X_2+h, t)+\theta(X_1, X_2-h, t)$$

$$-4\theta(\bar{X}, t)]+f(\theta(\bar{X}, t)), \quad (22)$$

where $\theta(\bar{X}, t)$ represents the dynamic variable, D is a positive constant proportional to the phenomenological diffusion coefficient, $f(\theta(\bar{X}, t)) = \theta(\bar{X}, t+\tau)$ is the single cell dynamic function. The function $f(\theta)$ is the solution semigroup corresponding to the ordinary differential equation $\mathrm{d}\theta/\mathrm{dt} = W(\theta)$ [17]. From this it follows that

$$\theta(\bar{X}, t+\tau) = \int_0^\infty W(\bar{X}, t)\,\mathrm{d}t = f(\theta) \quad (23)$$

$$F(\bar{X}, t) = \int_t^{t+\tau} W(\bar{X}, t)\,\mathrm{d}t \approx \tau W\left(\bar{X}, t+\frac{\tau}{2}\right) \quad (24)$$

$$f(\theta(\bar{X}, t)) = F(\bar{X}, t)+\theta(\bar{X}, t). \quad (25)$$

Relation (25) shows that at $D = 1$ discrete equations (20) and (22) coincide. Note that, for transfer processes, the phenomenological constant $D \equiv 1$, and the diffusion coefficient (thermal diffusivity) is determined by the relationship between τ and h, as shown below. Discrete transfer equations (20) and (22), as well as the discrete analogue of the Fourier law (21), can be employed for the analysis of transfer processes directly in discrete form, which is particularly convenient for numerical simulation, since equations (20)–(22) do not require translation into the language of discrete mathematics. Moreover, the discrete model allows one to describe a wider class of transfer processes than the parabolic-type classical equations, since it does not rely on the space–time locality principles.

5.1. *The relationship between the discrete model and partial differential equations*

In order to go from discrete equations to their partial differential approximations it is necessary to expand the functions $T(\bar{X}, t)$, $W(\bar{X}, t)$ and $\bar{q}(\bar{X}, t)$ in equations (20) and (21) into a Taylor series in powers of τ and h (for simplicity, a one-dimensional case is considered). After some transformations, equations (20), (21) and (24) yield

$$C_p\rho\left(\frac{\partial T}{\partial t}+\frac{\tau}{2}\frac{\partial^2 T}{\partial t^2}+\frac{\tau^2}{6}\frac{\partial^3 T}{\partial t^3}+\ldots\right)$$

$$= C_p\rho\left(\frac{h^2}{2\tau}\frac{\partial^2 T}{\partial X^2}+\frac{h^4}{\tau 4!}\frac{\partial^4 T}{\partial X^4}+\frac{h^6}{\tau 6!}\frac{\partial^6 T}{\partial X^6}+\ldots\right)$$

$$+W+\frac{\tau}{2}+\frac{\tau^2}{8}\frac{\partial^2 W}{\partial t^2}+\ldots \quad (26)$$

$$q+\frac{\tau}{2}\frac{\partial q}{\partial t}+\frac{\tau^2}{8}\frac{\partial^2 g}{\partial t^2}+\ldots$$

$$= -\frac{hC_p\rho}{2\tau}\left(h\frac{\partial T}{\partial X}+\frac{h^3}{12}\frac{\partial^3 T}{\partial X^3}+\ldots\right). \quad (27)$$

Partial differential equations (26) and (27) contain an infinite number of expansion terms. In order to get from (26) and (27), the equations with a finite number of terms, one should assign the limiting transition law, i.e. the relationship between τ and h for τ, $h \to 0$.

5.2. *The diffusional law of limiting transition*

Usually when going from discrete transfer equations to partial differential ones it is assumed that the thermal diffusivity (diffusion) coefficient $a = H^2/\tau$ remains a finite quantity when τ, $h \to 0$ (see refs. [14, 15] and references therein). Such a law of the limiting transition will be called 'diffusional'. Note that in this case the speed of heat signal (thermal disturbance) is $c = h/\tau = a/h \to \infty$. Assuming further that $\lambda = h^2 C_p\rho/2\tau$, from equations (26) and (27) we obtain in zero approximation with respect to τ the classical relationships of the local-equilibrium thermodynamics, which are parabolic-type transfer equations (diffusion equation) and the Fourier law.

In the first approximation with respect to τ, equations (26) and (27) yield

$$C_p\rho\frac{\partial T}{\partial t}+C_p\rho\frac{\tau}{2}\frac{\partial^2 T}{\partial t^2}$$

$$= \lambda\frac{\partial^2 T}{\partial X^2}+\frac{\lambda h^2}{12}\frac{\partial^4 T}{\partial X^2}+W+\frac{\tau}{2}\frac{\partial W}{\partial t} \quad (28)$$

$$q+\frac{\tau}{2}\frac{\partial q}{\partial t} = -\lambda\frac{\partial T}{\partial X}-\frac{\lambda h^2}{12}\frac{\partial^3 T}{\partial X^3}. \quad (29)$$

The terms of equations (28) and (29) that contain the factor τ reflect the local non-equilibrium state (temporal non-locality) of transfer processes. Therefore, such terms are absent both in the classical locally equilibrium equations and in non-local equations (16) and (18). The terms that involve the factor h^2 describe the spatial non-locality and, therefore, similar terms are also present in equations (16) and (18) (the difference in the numerical factors is due to the different forms of the non-locality functions adopted in these models). Note that in contrast to non-local equation (18) heat conduction equation (26), which takes into account the space–time non-locality of the system, does not involve either a mixed temperature derivative, or a space derivative of the source function W. This is due to the fact that the internal energy of the system and the intensity of the heat source in the heat transfer model for a discrete medium are assumed to be independent of the coordinate in each individual discrete medium element of size h. In the mode of non-local heat transfer (Section 3), the quantities E and W can vary over the distances of the order of the non-locality scale h. It is precisely this which leads to the appearance of appropriate derivatives in heat conduction equation (18).

Subsequent approximations of discrete heat transfer equations (26) and (27) will contain higher-order expansion terms. However, with the 'diffusional' law of limiting transition all these approximations, like the classical locally equilibrium equations and first approximations to discrete systems, equations (28) and (29), determine an infinitely large speed of propagation of the heat signal c.

5.3. *The 'wave' law of limiting transition*

Now consider the 'wave' law of limiting transition [6, 7] when the propagation speed of the heat signal (heat wave) remains finite for τ, $h \to 0$, i.e. $c = h/\tau = \text{const} < \infty$. In this case the thermal diffusivity $a = h^2/\tau \to 0$.

In zero approximation with respect to τ, equations (26) and (27) give

$$C_p\rho \frac{\partial T}{\partial t} = W, \quad q = 0.$$

These relationships imply, in this approximation, that the heat does not propagate throughout the system at all, and its temperature is entirely determined by an external heat source, and, certainly, by initial conditions.

In a first approximation with respect to τ and with the 'wave' law of limiting transition, equations (26) and (27) yield the following hyperbolic-type equations:

$$C_p\rho \frac{\partial T}{\partial t} + C_p\rho \frac{\tau}{2} \frac{\partial^2 T}{\partial t^2} = C_p\rho c^2 \frac{\tau}{2} \frac{\partial^2 T}{\partial X^2} + W + \frac{\tau}{2} \frac{\partial W}{\partial t} \tag{30}$$

$$q + \frac{\tau}{2} \frac{\partial q}{\partial t} = -C_p\rho c^2 \frac{\tau}{2} \frac{\partial T}{\partial X}. \tag{31}$$

In accordance with the 'wave' law of limiting transition these equations determine the finite speed of heat signal propagation. Assuming

$$C_p\rho c^2 \frac{\tau}{2} = \lambda \quad \text{and} \quad \tau^* = \frac{\tau}{2}$$

it can be seen that equation (30) coincides with the equation of transfer in a media with heat memory, equation (11), which involves an exponential function of heat flux relaxation, equation (9). Equation (31) coincides with the Maxwell–Cattaneo law, equation (10). The subsequent approximations of discrete heat transfer equations (20) and (21) in the case of the 'wave' law of limiting transition will also yield partial differential equations with the finite speed of heat signal propagation.

Thus, while going from discrete transport equations to their different accuracy approximations in the form of partial differential equations, one should assign the law of limiting transition, i.e. the relationship between the space–time scales of the internal structure of a medium. The law of limiting transition determines the type of equations and, consequently, certain fundamental features of their solutions. Strictly speaking, the 'wave' law of limiting transition seems to be more justified, since it prescribes the finite speed of heat signal propagation which corresponds to both the random walk model itself and the physical sense of transfer processes in general. However, for sufficiently slow processes (see Section 2), when the characteristic speed of the process is small as compared to the speed of heat signal propagation, the 'diffusional' law of limiting transition may be employed. The fact that in the process of limiting transition discrete equation (20) gives hyperbolic-type transfer equation (30) or (11), i.e. the 'telegraph' equation, is a consequence of the discrete model not being on the principles of local equilibrium and spatial locality. It is precisely this which makes it possible to describe both locally equilibrium transfer processes and locally non-equilibrium or spatially non-local processes with the help of the discrete model, equations (20) and (21).

6. CONCLUSIONS

Equations of transfer in non-local media, namely, locally non-equilibrium media (media with memory), non-local media and media with discrete structure, significantly differ from the classical local transfer equations. In locally non-equilibrium systems (i.e. systems with temporal non-locality), when the medium possesses heat memory (Section 3) or has a discrete structure (Section 5), the corresponding transfer equations determine the finite speed of heat signal propagation. This property also persists in partial differential equations resulting from a discrete transfer equation in the case of the 'wave' law of limiting transition (Section 5.3). In non-local media (Section 4), as well as in the classical system, the heat signal propagates with an infinitely high speed. With the 'diffusional' law of limiting transition (Section 5.2), the discrete transfer equation yields partial differential equations which also determine an infinitely large speed of heat signal propagation.

In the limiting cases the generalized transfer equation for a medium with memory (8) can give both the classical parabolic-type transfer equation and the hyperbolic-type 'telegraph' equation with distributed sources (11). The latter also follows from the discrete transfer equation (20) obeying the 'wave' law of limiting transition. The equation of transfer in a medium with memory takes into account the local non-equilibrium state of the system, but at the same time presupposes its spatial locality. In non-local media the process of transfer is respectively described by non-local equations (see Section 4). Note that for non-local systems one should use the law of energy conservation either in integral form, equation (14), or in differential form, but with non-local terms taken into account, equation (17).

Transfer equations (20) and the interrelation between the heat flux and temperature (the generalization of the Fourier law) in a discrete form, equation (21), take into consideration both the local non-equilibrium state of the system and its spatial non-locality. These equations can be used directly in discrete form, which is especially convenient for the numerical simulation of transfer processes and the analysis of the chaotic dynamics of non-linear system. Moreover, after the corresponding expansion into the Taylor series in powers of τ and h discrete transfer equations (20) and (21) yield partial differential equations of different types. The type of these equations is determined by the law of limiting transition, i.e. by the interrelation between the temporal τ and spatial h scales of the internal structure of the medium. Of greatest interest is the 'diffusional' law of limiting transition, when the thermal diffusivity (diffusion) cofficient $a = h^2/\tau$ remains finite for τ, $h \to 0$, and the 'wave' law when the speed of heat wave (heat signal) propagation $c = h/\tau$ is finite, i.e. $c = \text{const} < \infty$, if τ, $h \to 0$. In the case of the 'diffusional' law of limiting transition, equations (20) and (21) give parabolic-type equations and, in particular, the classical heat conduction (diffusion) equation and the Fourier law, whereas with the 'wave' law they yield hyperbolic-type equations, for example the 'telegraph' equation (11) and the Maxwell–Cattaneo law, equation (10).

The non-local equations should be used for analysing transfer processes when the approximations, on which the classical theory is based (see Section 2), are not fulfilled. Such a situation can take place in different kinds of high-speed processes, for example when a normal zone propagates along a superconductor, during the propagation of the waves of phase transition, and also of detonation and combustion waves [7]. Application of supershort laser radiation also leads to non-local phenomena, e.g. in plasma [12], in solids [13], and in metals (see the refs. [5, 7]). The non-local equations also describe transfer processes in systems with a complex heterogeneous structure, such as polymers, capillary-porous media, liquid crystals, porous rocks (sandstone and limestone), heterogeneous catalysts, etc. The effect of local non-equilibrium state is especially perceptible at low temperatures in superconductors and superfluid liquids [2, 5–7]. The spatial non-locality is characteristic for ecological systems in which the biomass exists in the form of separate discrete cells, species, populations, etc.

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